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BRANCHED TWO-SHEETED COVERS

BY

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ABSTRACT

In this paper we explore the connection between Weierstrass points of subspaces of the holomorphic differentials and the geometry of the canonical curve in $\mathbf{PC}^{g-1}$. In particular, we consider non-hyperelliptic Riemann surfaces with involution and the Weierstrass points of the -1 eigenspace of the holomorphic differentials. The case of coverings of a torus is considered in detail.

1. Introduction

The current paper originated as an attempt to understand the following situation. If S is a non-hyperelliptic compact Riemann surface of genus 3, then the canonical curve representing S is a quartic in $\mathbf{PC}^2$. It is well known that such a curve has 28 bitangents; see, for example, [3]. Examples can be constructed where 4 of these bitangents pass through a common point and the 8 points of bitangency are the intersection of the curve with a homogeneous quadric.

It is easy to see that any non-hyperelliptic surface S of genus 3 which is a branched two-sheeted cover ρ of a torus X satisfies the above condition. In this introduction we shall sketch the proof of this assertion. In (the remainder of) this paper we generalize this result and hence offer a (possible) explanation of what really lies behind it.

If S is a non-hyperelliptic surface of genus 3 and S admits a conformal involution E, then it follows that E has precisely 4 fixed points. This is a consequence of the fact that on a surface of genus 3, a conformal involution can have either 0,

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4 or 8 fixed points. If the surface has an involution with 0 or 8 fixed points, then it is hyperelliptic. (If S had a fixed point free involution, then it would cover a surface of genus 2. An unramified two-sheeted cover of a surface of genus 2 must be hyperelliptic. This result is a special case of the theorem of [2]; see [4] for an elegant and elementary proof of this fact.)

Elementary considerations show that we can choose a basis for the holomorphic differentials on the surface S so that one of the basis elements (say, θ) is E-invariant and the other two basis elements (ω_1, ω_2) are anti-invariant under the involution. The divisor of the invariant differential is $P_1 \cdots P_4$, where the points P_j, $j = 1, \ldots, 4$, are the fixed points of the involution E. The non-hyperellipticity of S is enough to guarantee that there is no point on S where two linearly independent anti-invariant differentials vanish simultaneously (see Lemma 2); so that $f = \omega_1/\omega_2$ is an E-invariant function of degree 4 on S.

The fiber $f^{-1}(\alpha)$ over any point α on the sphere $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ is the divisor of an anti-invariant holomorphic differential and is therefore canonical and invariant. The map f has 12 branch points (counting multiplicities). A simple argument shows that f is branched at each fixed point P of E, the branch number of the map at a fixed point must be odd (because the order of the zero of an anti-invariant differential at a fixed point is even; see Lemma 1); so that the branch number of f at a fixed point of E must be 1 or 3. It thus follows that either f has branch number 3 at each of the 4 fixed points of E or f is branched at some non-fixed point Q of E. It follows (because of the invariance of f) that in the latter case f is also branched at $E(Q)$. From these simple observations, we see that the divisors $f^{-1}(\alpha)$ that contain multiple points (corresponding to branch values α) must be of the form (i) P^4 for a fixed point P of E, or (ii) $P_1^2P_2^2$ for two distinct fixed points P_1 and P_2 of E, or (iii) $P^2QE(Q)$ with $E(P) = P$ and $E(Q) \neq Q$ or (iv) $Q^2E(Q)^2$ with Q not fixed by E.

Not all of the above possibilities can occur. If case (ii) occurred, then there would exist on S an anti-invariant differential ω with $(\omega) = P_1^2P_2^2$. Hence the divisor of the function θ/ω would be P_3P_4/P_1P_2 and S would be hyperelliptic contrary to hypothesis. It is also impossible for all the branching to be accounted for by case (iii); the maximum contribution to the total branch number for f from such branching is 4.

It follows from the above considerations that there are 4 distinct E-inequivalent points Q_j (some of these may be fixed points of E) with the property that the divisor $D_j = Q_j^2E(Q_j)^2 = f^{-1}(f(Q_j))$ is canonical and in fact the divisor of an anti-invariant holomorphic differential ω_{Q_j} on S. We shall show shortly that these 4 differentials determine the 4 bitangents to the canonical curve which pass through a common point. Such a bitangent is degenerate and a tangent when a Q_j is a fixed point of E. We embed the surface S into projective space $\mathbf{PC}^2$ using the holomorphic differentials of the first kind (to obtain the canonical (model of the) curve). Perfectly good affine coordinates for the image of a point $P \in S$ are given as $(\theta(P), \omega_1(P), \omega_2(P))$.† Let (z_0, z_1, z_2) denote the usual affine coordinates on $\mathbf{PC}^2$. Choose constants a_j, b_j so that $\omega_{Q_j} = a_j\omega_1 + b_j\omega_2$. Then the hyperplane $a_j z_1 + b_j z_2 = 0$ intersects the canonical curve exactly in the divisor D_j and therefore corresponds to a bitangent which passes through the point (1,0,0).

We have almost completed our story. To continue, observe that the invariant function $\omega_{Q_1}/\omega_{Q_2}$ projects to a meromorphic function on X with divisor $\rho(Q_1^2)/\rho(Q_2^2)$. If we now normalize the torus X by letting $\rho(Q_1) = 0$, then the $\{\rho(Q_j), j = 2,3,4\}$ are the other three points of order two. This follows from the fact that $\omega_{Q_j}/\omega_{Q_1}$ is an invariant function on S whose projection to X has divisor $\rho(Q_j)^2/\rho(Q_1)^2$. Finally, consider the invariant function on S defined by $\omega_{Q_1}^2/\theta^2$. It projects to a meromorphic function with divisor $\rho(Q_1)^4/\rho(P_1)\cdots\rho(P_4)$. From this we conclude that the divisors $\rho(P_1)\cdots\rho(P_4)$ and $\rho(Q_1)\cdots\rho(Q_4)$ are equivalent. Thus the divisors $Q_1\cdots Q_4E(Q_1)\cdots E(Q_4)$ and $P_1^2\cdots P_4^2$ on S are also equivalent and this shows that the former is the divisor of a holomorphic quadratic differential on S. The last statement is equivalent to the assertion in the first paragraph that the 8 points of bitangency lie on a homogeneous quadric.

In the general case, we are studying a closed non-hyperelliptic surface S of genus $g > 2$ that is a ramified covering of a compact surface X of genus $p \geq 1$ and view S as the canonical curve sitting in $\mathbf{PC}^{g-1}$. There exists a holomorphic involution E on S with $X = S/\langle E\rangle$. The involution also acts on $\mathbf{PC}^{g-1}$ yielding two non-intersecting invariant subspaces $\mathbf{PC}^{p-1}$ and $\mathbf{PC}^{g-p-1}$. On each of these spaces E acts as the identity. From the Weierstrass points for the $(g-p)$-dimensional space of E anti-invariant holomorphic differentials of the first kind on S, we construct a finite non-empty collection of *special pairs* of points on the surface S. There are at most $p(g-p)^2$ special pairs.‡ Each special pair of points determines a finite-dimensional family of hyperplanes in $\mathbf{PC}^{g-1}$. Each hyperplane intersects the curve S at the special points with high multiplicity (at least $g-p$). The hyperplane generally also intersects the curve at $2p-2$ other points. All the hyperplanes contain the fixed $\mathbf{PC}^{p-1}$ mentioned earlier.

†When discussing projective embeddings, we will often identify a differential $\omega = h(z)\,dz$ on S with the function h that represents it in terms of the local coordinate z. This will not cause any damage as long as we use the same local coordinate for all the differentials used to embed the surface.

‡Sharp estimates for the minimum number of special pairs are quite delicate. This topic is currently under study by the authors.

The general situation simplifies considerably for $p = 1$. In this case, the dimension of each family of hyperplanes is zero (each family consists of a single hyperplane) and there are exactly $(g - 1)^2$ pairs of special points. These pairs may degenerate in the sense that the two points in the pair coincide. Each hyperplane intersects the curve at precisely the special points with multiplicity $g - 1$ when the pair is non-degenerate. We obtain characterizations of the divisor of special points in terms of the Jacobian variety of and the Riemann theta function on X.

Our work is related to a paper of Riera and Rodriguez [4] who study the case $g = 2$ and $p = 1$. These authors call attention to some work of Poincaré on the same subject and promise to consider $g = 3$ and 4 in a future paper. Whereas [4] considers in detail the Fuchsian uniformization of genus 2 surfaces with nontrivial involutions (in addition to the hyperelliptic involution), the present work explores the geometry of the canonical curve of non-hyperelliptic surfaces (of arbitrary genus) with involutions.

2. Branched two-sheeted covers

We shall be studying two-sheeted holomorphic covers

$$\rho : S \to X$$

of Riemann surfaces. To fix notation, we let S and X be compact Riemann surfaces of genus g and p, respectively, ρ a degree-two holomorphic map between them, branched at the $2k$ points $P_1, P_2, \ldots, P_{2k}$. Riemann–Hurwitz tells us that $g = 2p + k - 1$. We define the *ramification divisor of* ρ by $D_\rho = \rho(P_1)\rho(P_2)\cdots\rho(P_{2k})$. The surface S admits a conformal involution, E, which satisfies $\rho \circ E = \rho$ and fixes the branch points P_i, $i = 1, \ldots, 2k$.

The points of order two in $J(X)$, the Jacobian variety of the Riemann surface X, classify the possible branched two-sheeted covers S of X which have the given ramification divisor. For the convenience of the reader, we proceed to describe this well-known classification. If $p = 0$ (then the surface S is hyperelliptic and) we regard $J(X)$ to be a point.

Let k be a non-negative integer and choose $2k$ distinct points $x_1, x_2, \ldots, x_{2k}$ on X. Let $D = x_1 \cdots x_{2k}$. The degree-two covers of X with ramification divisor D are in one-to-one correspondence with certain index-two subgroups of $\Pi_1(X - D)$, the fundamental group of the Riemann surface X punctured at the points in the ramification divisor D.

Let $\{\gamma_1, \ldots, \gamma_p; \delta_1, \ldots, \delta_p\}$ be a canonical homotopy basis for X based at some point x_0 that avoids the points x_j, $j = 1, \ldots, 2k$, and let c_j, $j = 1, \ldots, 2k$, be a path emanating from x_0 surrounding the point x_j and returning to the point x_0. We have described generators for the fundamental group of the surface X punctured at the points in the ramification divisor. This group, $\Pi_1(X - D)$, is generated by the above $2p + 2k$ paths subject to the single defining relation

$$\prod_{i=1}^{p} [\gamma_i, \delta_i] \prod_{j=1}^{2k} c_j = 1,$$

where $[\gamma, \delta] = \gamma\delta\gamma^{-1}\delta^{-1}$ is the commutator of γ and δ. In order to describe all the smooth degree-two covers of $X - D$, it is sufficient to find all index-two subgroups of the group generated by these paths, and therefore to find all homomorphisms of this group onto the group with two elements $\mathbf{Z}_2$. The kernels of these homomorphisms are the desired subgroups.

Each homomorphism is determined by its action on the generators and if we want to have branching at each of the $2k$ points in D, we need that the image of each c_j be 1. The images of the γ_i and δ_i can be arbitrary (we are using the fact that $2k$ is even). We can therefore associate with the homomorphism h from the fundamental group of $X - D$ to the group of order 2, the symbol

$$\begin{bmatrix} h(\gamma_1), \ldots, h(\gamma_p) \\ h(\delta_1), \ldots, h(\delta_p) \end{bmatrix}.$$

This symbol can be identified with the point of order 2 in $J(X)$, $h_\rho = \frac{1}{2}\sum_{i=1}^{i=p} (h(\delta_i)e^i + h(\gamma_i)\pi^i)$, where e^i and π^i are the respective i-th columns of the identity matrix and the period matrix π (whose (i,j)-entry is $\pi_{ij} = \int_{\delta_i} \theta_j$, where $\{\theta_i; i = 1, \ldots, p\}$ is the basis of the holomorphic differentials dual to the given canonical homology basis). For the sake of definiteness we use the same point x_0 as both a base point for the fundamental group of $X - D$ and for embedding X into its Jacobian variety.

Function theoretically, we are constructing the cover S on which a certain multivalued meromorphic function f on $X - D$ becomes single-valued. The function has a square root singularity at each point of the ramification divisor and changes sign over the appropriate paths according to the homomorphism h. Existence of such a function clearly leads to a two-sheeted cover S. Conversely, the existence of S defines such a multivalued function f as we shall see in the next section (where we also derive the connection between the function f and the point $h_\rho \in J(X)$).

3. Anti-invariant differentials

We continue with the situation treated in the previous section; that is, S is a closed Riemann surface of genus $g = 2p + k - 1$ with a conformal involution E

with $2k$ fixed points (thus the quotient surface $X = S/\langle E \rangle$ has genus p). It is well known that the vector space $\mathfrak{A} = \mathfrak{A}(S)$ of holomorphic differentials on S decomposes into the direct sum of two subspaces; the E-invariant subspace which we shall denote by $\mathfrak{A}^+ = \mathfrak{A}_E^+ = \mathfrak{A}_E^+(S)$ and the E-anti-invariant space which we shall denote by $\mathfrak{A}^- = \mathfrak{A}_E^- = \mathfrak{A}_E^-(S)$.

The space $\mathfrak{A}^+$ has (complex) dimension p and consists of the lifts to S of the holomorphic differentials on X. Each $\theta \in \mathfrak{A}^+$ vanishes at each of the fixed points P_i of E. Moreover, the vanishing is always to odd order. In fact, we have the following

Lemma 1. *Let ω be an anti-invariant (respectively, invariant) meromorphic differential on S and let P be a fixed point of E, then* $\operatorname{ord}_P \omega$ *is even (odd).*

Proof. Choose a local parameter z vanishing at P so that in terms of this coordinate, E is represented by $z \mapsto -z$. Let ω be represented by $w(z)\,dz$. If ω is anti-invariant under E, then $w(E(z))E'(z) = -w(z)$. This means that $w(-z) = w(z)$ (that is, w is an even function of the local coordinate z) and shows that ω has even order at any fixed point of E. The argument for invariant differentials is similar.

Remark. Let $D = D_\rho = x_1 \cdots x_{2k}$ be the ramification divisor of the natural cover $\rho : S \to X$, where $x_i = \rho(P_i)$. At this point we can construct a multivalued function on $X - D$ that determines the cover ρ. If X is of genus 0 and $x_i \neq \infty$ for all i, then we can use the function

$$f(z) = \sqrt{\prod_{i=1}^{2k} (z - x_i)}, \qquad z \in \hat{\mathbf{C}}.$$

If one of the x_i equals infinity, then the corresponding term is omitted from the above product. If $p > 0$, we let θ be a non-trivial invariant holomorphic differential on S and let ω be a non-trivial anti-invariant holomorphic differential on S (we have excluded the case $p = 1$ and $k = 0$ (in which case $g = 1$ also), which is left to the reader). The anti-invariant meromorphic function $f = \theta/\omega$ on $S - \rho^{-1}(D)$ projects to a multivalued function F on $X - D$ that defines the cover. We must check that continuing F along curves leads to the defining homomorphism h for the cover. It is obvious from the Lemma that F has square root singularities at the branch values of ρ and hence continuation of F along the curves c_i, $i = 1, \ldots, 2k$ (defined in the previous section) leads to a change of sign. The anti-invariance of the function f on S shows that F changes sign after continuation along a closed path on $X - D$ if and only if the path lifts to an open path in S (that is, if and only if the homomorphism h has value 1 on this closed path). One must also check that the homomorphism h is independent of the choices made. If one chooses arbitrary non-trivial $\theta_1 \in \mathcal{Q}_E^+$ and $\omega_1 \in \mathcal{Q}_E^-$, then the ratio f_1/f, where $f_1 = \theta_1/\omega_1$, is E-invariant and projects to a single-valued function on X. Hence F_1, the projection of f_1 to X, and F induce the same homomorphism h from $\Pi_1(X - D)$ to $\mathbf{Z}_2$. We will identify the homomorphism h with the point $h_\rho \in J(X)$ by Lemma 4.

An interesting invariant attached to a subspace of the space of holomorphic differentials on a compact Riemann surface is the set of *Weierstrass points* of the subspace. These are defined as the zeros of the Wronskian of any basis for the subspace. Let d be a positive integer. It is easy to see that for a d-dimensional space $\mathcal{B}$ of holomorphic differentials on S, a point P is a Weierstrass point for $\mathcal{B}$ if and only if there exists a $\theta \in \mathcal{B}$ that vanishes at P to order at least d. Let $\theta_1, \ldots, \theta_d$ be a basis for $\mathcal{B}$. We shall denote the Wronskian for this basis by

$$W = W(\theta_1, \ldots, \theta_d) = \det \begin{bmatrix} \theta_1 & \cdots & \theta_d \\ \vdots & \vdots & \vdots \\ \theta_1^{(d-1)} & \cdots & \theta_d^{(d-1)} \end{bmatrix}.$$

In the above we have identified the differential θ with its expression $\theta(z)\,dz$ in terms of a local coordinate z. It should be checked that W is a holomorphic $d(d+1)/2$ differential and hence its degree is $d(d+1)(g-1)$. This latter number is precisely the number of Weierstrass points, counting multiplicities, of $\mathcal{B}$. For any $P \in S$, $\operatorname{ord}_P W$ is the weight of the Weierstrass point P (see [1, III.5.8] for the definition).

Assume from now on that $p > 0$ and if $p = 1$, that $k > 0$; it follows that $g \geq 2$. We turn our attention now to the Weierstrass points of $\mathcal{Q}^-$. This space has (positive, because of the above restriction) dimension $d = g - p = p + k - 1$ and hence the number of Weierstrass points of $\mathcal{Q}^-$, counting multiplicities, is $(g-p)(g-p+1)(g-1) = (p+k-1)(p+k)(2p-2+k)$.

LEMMA 2. (a) *Each fixed point of E is a Weierstrass point of $\mathcal{Q}^-$ of weight at least $(g-p)(g-p-1)/2 = (p+k-1)(p+k-2)/2$.*

(b) *For each $P \in S$, there is an $\omega \in \mathcal{Q}^-$ that does not vanish at P except when S is hyperelliptic and $k = 0$ or 1.*

(c) *If S is hyperelliptic, $k = 0$ or 1, and H is the hyperelliptic involution on S, then every element of $\mathcal{Q}^-$ vanishes at the $4 - 2k$ fixed points of the involution $E \circ H$.*

(d) *Both $2g-2$ and $2g-4$ cannot be orders of zeros at a fixed point P of E of elements of $\mathfrak{A}^-$. Hence the weight of P with respect to the space $\mathfrak{A}^-$ is at most* $2(g-1)(d-1)-\frac{3}{2}d(d-1)+2$.

Proof. We begin with (b). Every E-invariant differential necessarily vanishes at each fixed point of E. It thus follows that if P is a fixed point of E, not all anti-invariant differentials can vanish at P. If P is not a fixed point of E, and if all anti-invariant differentials vanish at P, they would also have to vanish at $E(P)$. Furthermore, we can find a $p-1$ dimensional subspace of invariant differentials which will also vanish at P and $E(P)$. This implies that we have a $2p+k-2=g-1$ dimensional space of holomorphic differentials on S vanishing at the points P and $E(P)$. This of course implies that S is hyperelliptic and that $E(P)=H(P)$, where H is the hyperelliptic involution on S.

Let us assume now that S is hyperelliptic as above. We have seen that the involution $E\circ H$ fixes the point P. The action of H on $\mathfrak{A}$ is as minus the identity. It follows that $\mathfrak{A}_E^+=\mathfrak{A}_{E\circ H}^-$ and $\mathfrak{A}_E^-=\mathfrak{A}_{E\circ H}^+$. Since the order of an invariant differential at a fixed point of an involution ($E\circ H$, in this case) must be odd, every element of $\mathfrak{A}_E^-$ must vanish at each fixed point of $E\circ H$. We must determine which hyperelliptic surfaces can occur. The Lefschetz fixed point formula tells us that the number of fixed points of $E\circ H$ is $2(1-\operatorname{tr} E\circ H)=2(2-k)$. Thus $k=0$, 1 or 2 are the only possibilities. The case $k=2$ is eliminated because $E\circ H$ must have fixed points. Thus we are exactly in the excluded situation of the lemma. This proves parts (b) and (c) of the lemma.

In order to obtain (a), we observe that as a consequence of Lemma 1, the order of vanishing of an element of $\mathfrak{A}^-$ at a fixed point of E is even. It thus follows that the lowest possible orders of vanishing are $0,2,4,\ldots,2(p+k-2)=2(d-1)$. For part (d), assume that there existed anti-invariant holomorphic forms ω_1 that vanished at P to order $2g-2$ and ω_2 that vanished to order $2g-4$. Then the E-invariant function ω_2/ω_1 would be of degree 2 (its only pole would be at P and then this pole would be of order 2). This function defines a degree 1 function on X, contradicting the fact that X has positive genus.

Now to compute the highest possible weight of the fixed point P, we note that the largest collection of orders of zeros at this point of anti-invariant differentials are

$$0,2(g-d),2(g-d+1),\ldots,2(g-3),2(g-1). \tag{1}$$

Subtracting from this sum the sum of the lowest possible orders $(d(d-1)/2)$, we obtain the maximum possible weight of the Weierstrass point P for $\mathfrak{A}^-$.

REMARKS. (1) For $k > 2$, each fixed point P of E is a classical Weierstrass point on S (that is, a Weierstrass point for $\mathfrak{A}$) since there exists on S an element of $\mathfrak{A}^- \subset \mathfrak{A}$ which vanishes at P to order at least $2(p + k - 2) > g - 1$. This is a well-known fact. See, for example, [1, Theorem V.1.7].

(2) Each element of $\mathfrak{A}^+$ vanishes at the fixed points of E. It is obvious that for each $Q \in S$ that is not fixed by E, there is a $\theta \in \mathfrak{A}^+$ that does not vanish at Q (we lift an element of $\mathfrak{A}(X)$ that does not vanish at $\rho(Q)$).

(3) Each of the excluded cases actually occurs. Consider the hyperelliptic curve S of genus g given by

$$w^2 = \prod_{i=1}^{g+1} (z^2 - e_i^2),$$

where the $e_1^2, \ldots, e_{g+1}^2$ are $g + 1$ distinct non-zero complex numbers. The hyperelliptic involution H is given by the map $(z, w) \mapsto (z, -w)$ and the involution $E \circ H$ is given by $(z, w) \mapsto (-z, w)$. The quotient of S by $E \circ H$ is the curve

$$w^2 = \prod_{i=1}^{g+1} (z - e_i^2),$$

of genus $[g/2]$ (here $[\cdot]$ is the "integral part" function) and hence $E \circ H$ has 4 (2) fixed points when g is odd (even). If we now consider the involution E given by $(z, w) \mapsto (-z, -w)$ it will have no fixed points when g is odd and two fixed points when g is even.

(4) For $p = 1$ the result of (d) is sharp. We shall see after the proof of Lemma 6 that for $p > 1$ it is necessary that $k = 1$ in order for the upper bound to be attained. Even in this case there will be further restrictions.

The divisors of differentials in $\mathfrak{A}^-$ or $\mathfrak{A}^+$ are E-invariant. A Weierstrass point Q of $\mathfrak{A}^-$ will be called *special* if there exists a differential (the space of such differentials may have dimension bigger than 1) $\omega_Q \in \mathfrak{A}^-$ whose divisor is of the form $Q^{p+k-1}E(Q)^{p+k-1}\Delta E(\Delta)$, where Δ is an integral divisor of degree $p - 1$. Note that we do not exclude the possibility of Q appearing also in the support of Δ or $E(\Delta)$. We will call ω_Q a *special* differential corresponding to the point Q. Every Weierstrass point of $\mathfrak{A}^-$ which is not a fixed point of E is special; fixed points of E may or may not be special. There *must* exist special Weierstrass points. Otherwise, only the fixed points of E would be Weierstrass points and each would be of minimal weight. The total weight of the Weierstrass points would then be

$$(p + k - 1)(p + k - 2)k = (p + k - 1)(p + k)(2p - 2 + k),$$

which would imply that $p + k = 1$, a case we have excluded.

Lemma 3. *Let Q be a special Weierstrass point for $\mathfrak{A}^-$ and assume that Q is not a fixed point of E. Then it is not possible for there to exist anti-invariant differentials with zeros at Q of orders $g-2$ and $g-1$. Hence the maximum weight of such a point (with respect to $\mathfrak{A}^-$) is at most $d(p-1)+(2-p)$.*

Proof. The proof is similar to the proof of the previous lemma and uses the fact that for an anti-invariant differential ω, we have $\mathrm{ord}_Q\,\omega = \mathrm{ord}_{E(Q)}\,\omega$. The details are left to the reader.

Remark. The lemma is clearly sharp for $p=1$. See also Remark 4 above.

For $Q \in S$, we define τ_Q to be the weight of the point Q with respect to the space of anti-invariant differentials ($\tau_Q = \mathrm{ord}_Q\,W^-$, where W^- is the Wronskian for $\mathfrak{A}^-$). If Q is a Weierstrass point for $\mathfrak{A}^-$, then so is $E(Q)$. If Q is not fixed by E, then

$$\tau_Q + \tau_{E(Q)} \leq 2d(p-1) + 2(2-p).$$

Similarly, for a fixed point P of E, we have

$$\tau_P - \tfrac{1}{2}d(d-1) \leq 2(g-1)(d-1) - 2d(d-1) + 2 = 2d(p-1) + 2(2-p).$$

If Q_1 and Q_2 are two distinct special Weierstrass points of $\mathfrak{A}^-$ or if $Q_2 = E(Q_1)$ or if $Q_1 = Q_2$ and we choose (if possible) ω_{Q_1} not to be a multiple of ω_{Q_2}, then $\omega_{Q_1}/\omega_{Q_2}$ is a non-constant invariant function on S and therefore projects to a function on X. Its divisor on X will be $\rho(Q_1)^{p+k-1}\rho(\Delta_1)/\rho(Q_2)^{p+k-1}\rho(\Delta_2)$, with the obvious meaning for Δ_i, $i=1,2$. By Abel's theorem, the image of this divisor in the Jacobian variety of X is the point zero. Hence the divisors $\rho(Q_i)^{p+k-1}\rho(\Delta_i)$, with the Q_i ranging over the special Weierstrass points of $\mathfrak{A}_E^-$, are equivalent. We proceed to identify their image in the Jacobian variety $J(X)$.

4. The embedding into the Jacobian variety

We continue with the notation of the previous section and concentrate on the quotient surface $X = S/\langle E\rangle$. Fix a point $x_0 \in X$ and let $\Phi = \Phi_{x_0}$ denote the map of X into its Jacobian variety $J(X)$ via integration from the base point x_0. Normally one studies the map Φ from divisors of degree zero on X into the Jacobian variety $J(X)$. We need to consider a more general class of divisors on X of the form $\prod_{j=1}^n y_j^{\alpha_j}$, where as usual $y_j \in X$, but we permit α_j to be a half-integer whenever y_j is a point in the ramification divisor D and require α_j to be an integer otherwise. We also require that $\sum_{j=1}^n \alpha_j = 0$. The divisor (F) of a multivalued function F that defines the cover $\rho : S \to X$ is precisely of this type. We must exercise some care in defining the map Φ on these more general divisors.

We represent our surface X as a $4p$-sided polygon $\mathcal{P}$ with identifications. The sides of $\mathcal{P}$ represent the curves in the canonical homotopy basis $\{\gamma_1, \ldots, \gamma_p; \delta_1, \ldots, \delta_p\}$ for X with the vertices of $\mathcal{P}$ representing the base point x_0 (we select one of these vertices and consider it as a base point for integration). We choose the polygon so that the points $\{x_1, \ldots, x_{2k}\}$ in the ramification divisor are in the interior of $\mathcal{P}$ and hence so are the curves $\{c_1, \ldots, c_{2k}\}$ around these punctures on $X - D$. When integrating a differential on a path from the base point x_0 to a point x_j, $j = 1, \ldots, 2k$, we will insist that the curve along which we integrate stays inside $\mathcal{P}$ except for the end point x_0.

LEMMA 4. *Let F be a multivalued function defining the cover ρ and h_ρ the point of order 2 in $J(X)$ corresponding to the cover ρ. Then we have $\Phi((F)) = h_\rho$.*

PROOF. We adopt the notation of §2. Let $(F) = \prod_{j=1}^n y_j^{\alpha_j}$, where $y_j \in X$, α_j is a half-integer whenever y_j is a point in the ramification divisor D and α_j is an integer otherwise with $\sum_{j=1}^n \alpha_j = 0$. The l-th component of the vector $\Phi((F))$ is given by $\sum_{j=1}^n \alpha_j \int_{x_0}^{y_j} \theta_l$. To evaluate this expression we note that

$$dF/F = \sum_{i=1}^{n} \alpha_i \tau_{y_i x_0} + \sum_{i=1}^{p} \beta_i \theta_i,$$

where τ_{yx}† is the unique abelian differential of the third kind on X with simple poles at y and x (and regular elsewhere), residue $+1$ at y, residue -1 at x and $\int_{\gamma_l} \tau_{yx} = 0$ for $l = 1, \ldots, p$ (this last condition must be interpreted on the curve rather than homotopy level) and β_i is a complex constant for $i = 1, \ldots, p$.

We begin by evaluating both sides of the last equation. First,

$$\int_{\gamma_l} dF/F = \log F(\text{end point of } \gamma_l) - \log F(\text{initial point of } \gamma_l).$$

The initial and terminal points of the above integration are, of course, the same (on the surface X, not the polygon $\mathcal{P}$). However, on traversing the path γ_l the multivalued function F is multiplied by the constant $(-1)^{h(\gamma_l)}$. Here h is the homomorphism from the fundamental group of $X - D$ into $\mathbf{Z}_2$ defining the cover ρ. Hence we see that

$$\int_{\gamma_l} dF/F = \pi\iota(h(\gamma_l) + 2m_l),$$

†We are using the symbol τ_Q to denote weights of Weierstrass points (on S) and the symbol τ_{yx} to denote abelian differentials (on X). This abuse of symbols should not cause any confusion.

for some integer m_l. Similarly,

$$\int_{\delta_l} dF/F = \pi\iota(h(\delta_l) + 2n_l),$$

for some integer n_l. Second,

$$\int_{\gamma_l} \left(\sum_{i=1}^{n} \alpha_i \tau_{y_i x_0} + \sum_{i=1}^{p} \beta_i \theta_i \right) = \beta_l$$

and

$$\int_{\delta_l} \left(\sum_{i=1}^{n} \alpha_i \tau_{y_i x_0} + \sum_{i=1}^{p} \beta_i \theta_i \right) = \sum_{i=1}^{n} \alpha_i \int_{\delta_l} \tau_{y_i x_0} + \sum_{i=1}^{p} \beta_i \pi_{il} = 2\pi\iota \sum_{i=1}^{n} \alpha_i \int_{x_0}^{y_i} \theta_l + \sum_{i=1}^{p} \beta_i \pi_{il}.$$

The last equality is a consequence of the bilinear relations of Riemann. Comparing the two sets of integrals, we see that

$$\beta_i = \pi\iota(h(\gamma_i) + 2m_i), \qquad i = 1, \ldots, p,$$

and

$$\sum_{i=1}^{n} \alpha_i \int_{x_0}^{y_i} \theta_l = \frac{1}{2} \left((h(\delta_l) + 2n_l) - \sum_{i=1}^{p} (h(\gamma_i) + 2m_i)\pi_{il} \right).$$

This completes the proof of the lemma. The argument is a mixture of known methods. See, for example, [1, III.6.3 and III.9.15].

At this point we can give an alternate description of the space $\mathfrak{A}^-$ in terms of multivalued differentials. We begin by recalling that an E-invariant meromorphic function f on S corresponds to a meromorphic function F on X with $f = F \circ \rho$. If

$$(F) = \prod_{i=1}^{2k} x_i^{\alpha_i} \prod_{j=1}^{n} y_j^{\beta_j},$$

with $x_i \neq y_j$ for all pairs of indices i and j, the α_i are integers, the β_j are non-zero integers with

$$\sum_{i=1}^{2k} \alpha_i + \sum_{j=1}^{n} \beta_j = 0,$$

then

$$(f) = \prod_{i=1}^{2k} P_i^{2\alpha_i} \prod_{j=1}^{n} Q_j^{\beta_j} \prod_{j=1}^{n} E(Q_j)^{\beta_j},$$

where $\rho(Q_j) = y_j$. Similarly, an anti-invariant meromorphic function f on S is the lift of a multivalued meromorphic function F on X belonging to the character h on the fundamental group $X - D$. (A function F *belongs* to a character h if continuing F along the closed curve c on $X - D$ leads to $(-1)^{h(c)}F$.) Note that the nature of our character h forces F to have square root singularities at the points in the support of D. The relation between (F) and (f) is identical to the one given for the invariant case with the one important exception that each α_j is a half-integer and *not* an integer.

Now every element $\theta \in \mathfrak{A}_E^+(S)$ is the lift of a $\Theta \in \mathfrak{A}(X)$. If

$$(\Theta) = \prod_{i=1}^{2k} x_i^{\alpha_i} \prod_{j=1}^{n} y_j^{\beta_j},$$

with $x_i \neq y_j$ for all pairs of indices i and j, the α_i are non-negative integers, the β_j are positive integers with

$$\sum_{i=1}^{2k} \alpha_i + \sum_{j=1}^{n} \beta_j = 2p - 2,$$

then

$$(\theta) = \prod_{i=1}^{2k} P_i^{2\alpha_i+1} \prod_{j=1}^{n} Q_j^{\beta_j} \prod_{j=1}^{n} E(Q_j)^{\beta_j},$$

where $\rho(Q_j) = y_j$. Note that

$$\sum_{i=1}^{2k} (2\alpha_i + 1) + 2\sum_{j=1}^{n} \beta_j = 2(2p-2) + 2k = 2g - 2.$$

Finally, the space $\mathfrak{A}_E^-(S)$ is the lift of a space of holomorphic Prym differentials on the surface $X - D$ that belong to the character h and that are permitted to have half-order poles at the points in the ramification divisor. Again the relation between the divisors of the multivalued differential Ω on X and its lift ω to S is exactly as in the invariant case except that each α_j must be half-integer which is not an integer and at least $-1/2$. Of course, one may consider arbitrary multivalued differentials on X belonging to the character h. For these, we drop the condition on the bounds of the exponents α_i and β_j.

LEMMA 5. (a) *Let Ω be an arbitrary non-trivial meromorphic multivalued differential on X belonging to the character h. Then $\Phi((\Omega)) = -2K_{x_0} + h_\rho$, where h_ρ is the half-period in $J(X)$ determined by the cover and $-2K_{x_0}$ is the image in $J(X)$ of a canonical divisor on X. Conversely, if $\mathfrak{D}$ is a divisor on X of degree*

$2p-2$ with $\Phi(\mathfrak{D}) = -2K_{x_0} + h_\rho$, then there exists a meromorphic multivalued differential Ω on X belonging to the character h with $(\Omega) = \mathfrak{D}$.

(b) *Let $Q \in S$ be a special Weierstrass point for $\mathfrak{A}_E^-$ with ω_Q a corresponding differential. Write, as usual, $(\omega_Q) = Q^d \Delta E(Q)^d E(\Delta)$, where Δ is an integral divisor on S of degree $p-1$. Then*

$$\Phi(\rho(Q)^{p+k-1}\rho(\Delta)) = \tfrac{1}{2}\Phi(D_\rho) - 2K_{x_0} + h_\rho,$$

where D_ρ is the ramification divisor of ρ.

Proof. Let F be a multivalued function defining the cover ρ. An arbitrary multivalued meromorphic differential Ω on X belonging to the character h may be written as $\Omega = \varphi F$, with φ a meromorphic one form on X. The divisor $Z = (\varphi)$ is canonical and hence the first assertion of the lemma is a consequence of the previous lemma. The converse to part (a) is easily established. For the second assertion, we note that the projection of ω_Q to X has divisor $\rho(Q)^d \rho(\Delta) D_\rho^{-1/2}$.

Remarks. (1) One can (and we do) choose for K_{x_0}, the vector of Riemann constants for the base point x_0. See [1, Chapter VI].

(2) If $P \in S$ is a Weierstrass point for $\mathfrak{A}^-$, but P is not special, then we can find a $\omega_P \in \mathfrak{A}^-$ with $(\omega_P) = P^{p+k-2}\Delta E(P)^{p+k-2}E(\Delta)$, where Δ is an integral divisor of degree p with P *not* appearing in Δ (of course, $E(P) = P$). For these divisors we have

$$\Phi(\rho(P)^{p+k-2}\rho(\Delta)) = \tfrac{1}{2}\Phi(D_\rho) - 2K_{x_0} + h_\rho.$$

It is interesting to observe that one has a Riemann–Roch type theorem for the anti-invariant functions on S. We begin with some remarks. The order of an anti-invariant function at a fixed point of E is necessarily odd and the divisor of an anti-invariant function must be invariant under E. Every point in $\rho^{-1}(D_\rho)$, the lift to S of the ramification divisor, must necessarily appear (to a positive or negative power) in the divisor of a non-trivial anti-invariant function. This last remark is a consequence of the fact that for any anti-invariant function f, $f - f\circ E = 2f$; so that if the fixed point P of E is not a pole of f it is necessarily a zero of f.

Let a be an integral divisor on S. We decompose a into $a_s b$ where a_s is the largest (in the lexicographic ordering) E-invariant divisor less than or equal to a (which may be trivial) and b its complement relative to a. From our previous remarks it is clear that if we denote by $L_-[1/a]$ the vector space of anti-invariant functions whose divisors are multiplies of $1/a$ and if we denote the dimension of $L_-[1/a]$ by $r_-[1/a]$, then $r_-[1/a] = r_-[1/a_s]$.

The divisor a_s may have points of the preimage of the ramification divisor (that is, fixed points of E) in its support. Denote these by the divisor $P_{i_1}^{\alpha_1} \cdots P_{i_r}^{\alpha_r}$ where the α_j are positive integers. If any of the positive integers α_j is odd, increase it by one to obtain adjusted divisors a' and a_s'. Clearly, $a_s' = cE(c)$ for some integral divisor c on S and $r_-[1/a_s] = r_-[1/a_s']$.

Lemma 6. *Let a be an integral divisor on S. Then*

$$r_-[1/a] = r_-[1/a_s'] = \tfrac{1}{2}\deg(a_s') - (p + k - 1) + i_-[a_s'],$$

where $i_-[\Delta]$ is the dimension of the space of anti-invariant differentials whose divisors are multiples of Δ.

Proof. The first equality has already been verified in the remarks preceding the statement of the Lemma. It remains to explain the last equality. The vector space $L[1/a_s']$† has a canonical decomposition as the direct sum of subspaces of invariant and anti-invariant functions. The invariant functions in the space are precisely those which are the lifts of meromorphic functions from X and therefore their dimension is computed by the Riemann-Roch theorem (applied on X) to be

$$r[1/\rho(c)] = \tfrac{1}{2}\deg(a_s') - p + 1 + i[\rho(c)].$$

The Riemann-Roch theorem on S gives

$$r[1/a_s'] = \deg(a_s') - (2p + k - 1) + 1 + i[a_s'].$$

The difference between $r[1/a_s']$ and $r[1/\rho(c)]$ is $r_-[1/a]$. To obtain the second equality in the lemma, use the fact that $i[a_s'] = i_-[a_s'] + i_+[a_s']$ and $i_+[a_s'] = i[\rho(c)]$.

As an example we consider the divisor $a = P^{2l}$, where P is an arbitrary fixed point of E on S and l is a positive integer. The formula reads as follows:

$$r_-\left[\frac{1}{P^{2l}}\right] = l - (p + k - 1) + i_-[P^{2l}].$$

We have already remarked that the order of an anti-invariant function (differential) at a fixed point of E is necessarily odd (even) so that $r_-[1/P^{2l-1}] = r_-[1/P^{2l}]$ and $i_-[P^{2l-1}] = i_-[P^{2l}]$. It thus follows that

$$r_-\left[\frac{1}{P^{2l-1}}\right] = l - (p + k - 1) + i_-[P^{2l-1}].$$

†We use standard notation as in [1, Chapter III] except that we have substituted square brackets for parentheses.

Furthermore, it is also clear that if $l < 2k - 1$ there are no non-trivial anti-invariant functions in $L[1/P^{2l}]$ (because such a function must vanish at each of the $2k - 1$ other fixed points of E). These ideas are helpful in the computation of the weight of a Weierstrass point for the anti-invariant differentials.

The Weierstrass gap theorem on S asserts that there are precisely $g = d + p$ non-gaps bigger than 1 and less than or equal to $2g$. At a fixed point P of E, these non-gaps have the following property: the odd non-gaps are non-gaps for the anti-invariant functions and the even non-gaps are non-gaps for the invariant functions. This is seen by writing an arbitrary function as the sum of an invariant and an anti-invariant function. Since by Lemma 6, $r_-[1/P^{2g}] = p$, we find that in the sequence of odd integers in $\{1,2,\ldots,g\}$ there are precisely p non-gaps and d gaps for the anti-invariant functions. Therefore in the corresponding sequence of even integers there are precisely d non-gaps and p gaps for the invariant functions.

With the aid of Lemma 6 and the above observations, we can elaborate on our earlier remark concerning the sharpness of (d) in Lemma 2. How can the upper bound given in Lemma 2 be attained? We observed in the proof of Lemma 2 that the upper bound is attained when the orders of the zeros of the basis for $\mathfrak{A}^-$ adapted to the point P are given by (1) in §3. This implies that $i_-[P^{2(g-d)}] = d - 1$ which by Lemma 6 implies that $r_-[1/P^{2(g-d)-1}] = p - 1$. Recalling that $g - d = p$, we conclude that for $l = 2,\ldots,p$, $r_-[1/P^{2l-1}] = l - 1$ since there are only functions of odd degree in these spaces. It thus follows that among these $p - 1$ odd integers $2l - 1$ are non-gaps for the anti-invariant functions. For the remainder of this section "non-gap" means "at P and with respect to the space of anti-invariant functions." In particular, 3 is a non-gap which gives us an anti-invariant function of degree 3. We have already remarked that all the other fixed points of E are then necessarily zeros of this function; which allows us to conclude that $k \leq 2$.

We have observed above that there are precisely p odd non-gaps $\leq 2g - 1$ and we have thus far accounted for $p - 1$ of them. The maximum weight is attained when $2g - 3$ is the p-th and last non-gap (again by (1) and the fact that the orders of the zeros of the anti-invariant differentials are obtained by subtracting one from the list of gaps). To complete the picture, it remains to consider the (possible) cases $k = 1$ and $k = 2$. We assume that $p > 1$ since the case $p = 1$ is clarified by Lemma 9.

If $k = 1$, then $g = 2p$ and $2g - 3 = 4p - 3$. It thus follows that 3 and 5 are both non-gaps. Since the product of a triple of anti-invariant functions is anti-invariant, it follows that $9,11,13,\ldots,$ are all non-gaps as well. This leaves only 1 and 7 as odd gaps and by our previous remarks $d = 2$. Since in our case $d = p$, we have only the possibility of $p = 2$ and hence $g = 4$. There is, in fact, an example of such a surface. It is the Riemann surface of the algebraic curve

$$w^3 = z(z^2 - 1)(z^2 - \lambda^2), \qquad \text{with } \lambda \neq \pm 1.$$

This surface permits the involution $(w,z) \mapsto (-w,-z)$. The fixed points are the two points $z^{-1}(0)$ and $z^{-1}(\infty)$ whose weights are 5 and the remaining Weierstrass points for the anti-invariant differentials are the points lying over $1,-1,\lambda,-\lambda$ each of weight 2.

If $k = 2$, then we have $g = 2p + 1$. If also $p \geq 2$, we find, as before, that $d = 2$; which now however implies $2 = d = p + 2 - 1 = p + 1$ which is a contradiction.

5. The canonical model of the curve

Use the notation and conventions of the last two sections. We are now in a position to continue our investigation of the Weierstrass points of $\mathfrak{A}^-$. We have already seen in Lemma 2 that the fixed points $\{P_i; i = 1, \ldots, 2k\}$ of E are all Weierstrass points for the anti-invariant differentials and that the minimum weight of a fixed point is $d(d-1)/2$, where $d = p + k - 1 = \dim \mathfrak{A}_E^-$. Hence it makes sense to factor out from the divisor of Weierstrass points the appropriate power of the lift of the ramification divisor; namely, to factor out the divisor $(P_1 \cdots P_{2k})^{d(d-1)/2}$.

Lemma 7. *Let D_{W^-} denote the divisor of Weierstrass points of the anti-invariant differentials on S with each point repeated according to its multiplicity. Then we have*

$$D_{W^-} = (P_1 \cdots P_{2k})^{d(d-1)/2} \Delta E(\Delta),$$

where Δ is an integral divisor of degree pd^2.

Proof. It is clear that if Q is a zero of the Wronskian W^- of a given order then so is $E(Q)$. Thus the only part that needs verification concerns the order of W^- at a fixed point P of E. We compute

$$\operatorname{ord}_P W^- = \sum_{i=1}^{i=d} \operatorname{ord}_P \omega_i - d(d-1)/2,$$

where $\{\omega_i; i = 1, \ldots, d\}$ is a basis for $\mathfrak{A}^-$ adapted to the point P. Since each $\operatorname{ord}_P \omega_i$ is even, $\operatorname{ord}_P W^-$ differs from $d(d-1)/2$ by an even integer.

Before stating our first theorem we need to introduce one more ingredient. We use the canonical embedding of the surface S into $\mathbf{PC}^{g-1}$ (assuming that S is not hyperelliptic) and identify the surface with its canonical model. To be specific (for the purpose of having good coordinates), we shall use a basis for $\mathcal{A}$ to consist of $\{\theta_1,\dots,\theta_p;\omega_1,\dots,\omega_d\}$, where $\{\theta_1,\dots,\theta_p\}$ is a basis for $\mathcal{A}_E^+$ and $\{\omega_1,\dots,\omega_d\}$ is a basis for $\mathcal{A}_E^-$. There is a one-to-one correspondence between hyperplanes in projective space and abelian differentials of the first kind on the surface. The hyperplane **HP** $0 = \sum_{i=0}^{g-1} \lambda_i z_i$ is identified with the abelian differential of the first kind

$$\theta = \sum_{i=0}^{p-1} \lambda_i \theta_{i+1} + \sum_{i=p}^{p+d-1} \lambda_i \omega_{i-p+1}.$$

Thus the intersection of S with **HP** consists of the zeros of the differential θ. The *multiplicity* of an intersection point is the order of vanishing of the differential at the point of the curve. Similarly, the zeros in projective space of a homogeneous polynomial of degree q can be identified with a holomorphic q-differential on the surface.

To continue, we can identify $\mathbf{PC}^{g-1}$ with the projective space of (complex) lines $\mathbf{P}\mathcal{A}$ in the affine space $\mathcal{A}$ of holomorphic one forms on S. By duality, the points of projective space may also be seen as hyperplanes in $\mathbf{P}\mathcal{A}$. In this context the canonical map sends the point $Q \in S$ to the hyperplane $\Omega[Q]$ of differentials vanishing at Q. The involution E acts on $\mathbf{P}\mathcal{A}$ via its action on $\mathcal{A}$. In terms of the affine coordinates,

$$E(z_0,\dots,z_{g-1}) = (z_0,\dots,z_{p-1},-z_p,\dots,-z_{g-1}).$$

The involution E acts as the identity on the non-empty complimentary subspaces

$$\mathbf{PC}^{p-1} = \{(z_0,\dots,z_{g-1}) \in \mathbf{PC}^{g-1};\ z_p = 0 = \cdots = z_{g-1}\} = \mathbf{P}\mathcal{A}^+$$

and

$$\mathbf{PC}^{g-p-1} = \{(z_0,\dots,z_{g-1}) \in \mathbf{PC}^{g-1};\ z_0 = 0 = \cdots = z_{p-1}\} = \mathbf{P}\mathcal{A}^-.$$

THEOREM 1. (a) *The divisor of the Weierstrass points for $\mathcal{A}_E^-$ is*

$$D_{W^-} = (P_1 \cdots P_{2k})^{d(d-1)/2} Q_1 \cdots Q_{d^2p} E(Q_1) \cdots E(Q_{d^2p}),$$

where $Q_1,\dots,Q_{d^2p}$ are special Weierstrass points on S (and each special Weierstrass point is one of these or its image by E). Further, D_{W^-} is the divisor of an E-invariant (anti-invariant) holomorphic $d(d+1)/2$ differential on S when d is even (odd).

(b) *If $p = 1$, then $(d = k$ and$)$ the divisor $Q_1 \cdots Q_{k^2} E(Q_1) \cdots E(Q_{k^2})$ is k-canonical.*

(c) *Assume that S is not hyperelliptic and $p = 1$. In this case, for each i, the points Q_i and $E(Q_i)$ are the points of intersection of multiplicity k of a hyperplane in $\mathbf{PC}^{g-1}$ with the embedded surface. All these hyperplanes pass through the point* $(1,0,\dots,0)$.

(d) *For non-hyperelliptic S and general $p > 0$, the points Q_i and $E(Q_i)$ are the points of intersection of a hyperplane in $\mathbf{PC}^{g-1}$ with the embedded surface of multiplicity at least d. All these hyperplanes contain the subspace of $\mathbf{PC}^{p-1}$ spanned by the first p coordinates.*

Proof. Part (a) has already been established except for the invariance property of D_{W^-} which is easily checked.

To prove (b), we note that $P_1 \cdots P_{2k}$ is a canonical divisor on S (since it is the divisor of the lift to S of a non-trivial holomorphic one form on the torus X). Hence $D_{W^-}/(P_1 \cdots P_{2k})^{k(k-1)/2}$ is k-canonical.

For parts (c) and (d), we observe that under the canonical embedding a fixed point P of the automorphism E gets mapped to

$$(0,\dots 0,\ \omega_1(P),\dots,\omega_d(P)) \in \mathbf{PC}^{g-1}.$$

For each point Q_i, there is a holomorphic differential in $\mathfrak{A}^-$ with divisor $Q_i^d E(Q_i)^d \Delta E(\Delta)$, where Δ is an integral divisor of degree $(p-1)$. There exist complex numbers λ_j, $j = 1,\dots,d$, with $\sum_{j=1}^d \lambda_j \omega_j = \omega_{Q_i}$ and therefore the hyperplane $\sum_{j=1}^d \lambda_j z_{j+p-1} = 0$ intersects the embedded curve in the divisor of ω_{Q_i}. This proves (d) since the hyperplane contains the subspace spanned by the first p coordinates. In particular, when $p = 1$, the hyperplane passes through the point $(1,0,\dots,0)$.

Caution. The points $Q_1,\dots,Q_{d^2p}$ include all the special Weierstrass points but need *not* be distinct and some of these points might be fixed points of E (that is, it is possible for $P_i = Q_j$ for some pair of indices i and j). See below and also §6 and §7.

We are interested in computing the number N of distinct E-inequivalent points Q_i appearing in the divisor $D_{W^-}/(P_1 \cdots P_{2k})^{d(d-1)/2}$. We write this divisor as $\mathfrak{D}E(\mathfrak{D})$ (thus $\mathfrak{D} = Q_1 \cdots Q_{d^2p}$) and we want to determine the number of distinct points in $\mathfrak{D}$. By relabeling the indices we may assume that $Q_1,\dots,Q_N$ is the maximal list of distinct entries in $\mathfrak{D}$. We now recall the discussion following the proof of Lemma 3. We know that $\deg \mathfrak{D} = d^2p$ and that the point Q_j, $j = 1,\dots,N$, has multiplicity τ_{Q_j} in $\mathfrak{D}$ if it is not fixed by E and multiplicity $\frac{1}{2}(\tau_{Q_j} - \frac{1}{2}d(d-1))$ if it is fixed by E. It thus follows that

$$N \geq \frac{d^2 p}{d(p-1)+2-p} = \frac{(g-p)^2 p}{(g-p-1)(p-1)+1},$$

and hence for $p = 1$, $N = (g-1)^2$ and for $p = 2$, $N \geq 2(g-2)$.

We now define

$$\mu_Q + 1 = i_-[Q^d] = i_-[Q^d E(Q)^d]$$

for a special Weierstrass point Q that is not fixed by E and

$$\mu_P + 1 = i_-[P^{2d}]$$

for a special Weierstrass point P that is fixed by E. We need one more definition. A hyperplane intersects the curve *symmetrically* with respect to E if whenever R is a point of intersection of multiplicity α so is $E(R)$. Further, we require that the multiplicity of a fixed point of E in this intersection be even. This last condition is vacuously satisfied for $p = 1$. We have established part of the following

THEOREM 2. (a) *Let S be a non-hyperelliptic surface. Let $Q_1, \ldots, Q_N$ be the list of distinct special E-inequivalent Weierstrass points for $\mathfrak{A}^-$. Then each point Q in this list determines a μ_Q-dimensional family of hyperplanes that contain* $\mathbf{P}\mathfrak{A}^+$ *and intersect the canonical curve S at the points Q and $E(Q)$ each with multiplicity at least d. The other intersections of the curve and the hyperplane are of multiplicity at most $2p-2$. (Note that $d \geq 2p-2$ as soon as $g \geq 3p-2$.) Further, $N \geq d^2 p(d(p-1)+2-p)$. For $p = 1$, each $\mu_Q = 0$, there are no other points of intersection of the hyperplane and the curve, and we have equality in the formula for N and in the multiplicity of the intersection points.*

(b) *The special Weierstrass points account for all hyperplanes that intersect the curve symmetrically with respect to E and have at least one non-fixed point in the intersection that has multiplicity at least d or one fixed point in the intersection with multiplicity at least $2d$.*

PROOF. Part (a) has already been proven. For part (b), let **HP** be a hyperplane that intersects the curve symmetrically with high multiplicity. Then the corresponding differential θ has divisor

$$Q^d E(Q)^d \Delta^{p-1} E(\Delta)^{p-1}.$$

It follows that either $\theta \in \mathfrak{A}^+$ or $\theta \in \mathfrak{A}^-$. But θ cannot be in $\mathfrak{A}^+$ because it vanishes to even order (possibly order zero) at the fixed points of E. Thus θ is an ω_Q for a special Weierstrass point for the space of anti-invariant differentials.

CAUTION. In the statement of part (a) of the theorem, it may occur that Q is a fixed point of E. In this case there is a single point of intersection of multiplicity $2d$ instead of two points of multiplicity d.

There are relations between the numerical invariants associated to the special Weierstrass points. For example, it is easy to establish the following

PROPOSITION 1. *If Q is not a fixed point of E, then*

$$1 \leq (\mu_Q + 1)^2 \leq \tau_Q.$$

PROOF. The equality $i_-[Q^d] = s$ implies that the orders of vanishing at Q of the last s differentials in a basis for $\mathfrak{A}^-$ adapted to Q are at least $d, d+1, \ldots, d+s-1$. The result follows easily from this observation.

REMARK. If we consider the particular case $g = 3$ and $k = 2$, then the content of our last two theorems is that the canonical curve corresponding to the compact Riemann surface S has 4 bitangents which pass through a common point and the eight points of bitangency $Q_1, \ldots, Q_4, E(Q_1), \ldots, E(Q_4)$ are the zeros of a holomorphic quadratic differential on S. It was precisely this result which was the motivation for this paper as explained in the introduction.

In the next section we shall identify the image in X under ρ of the divisor $Q_1 \cdots Q_{d^2p} E(Q_1) \cdots E(Q_{d^2p})$. More accurately, we will identify the points $\rho(Q_i)$ on X and the divisor $\rho(Q_1) \cdots \rho(Q_{d^2p})$ of degree d^2p on X. We have already seen in the proof of Lemma 5 that for any base point x_0 on X, there is an integral divisor Δ_i (on S) of degree $(p-1)$, such that

$$\Phi_{x_0}(\rho(Q_i)^d \rho(\Delta_i)) = \tfrac{1}{2}\Phi_{x_0}(D_\rho) - 2K_{x_0} + h_\rho.$$

We shall characterize the relevant points on the Jacobian variety of X in terms of the Riemann theta function on X.

6. Theta functions on Riemann surfaces

If X is a compact Riemann surface of genus $p \geq 1$ together with a canonical homology basis $\{\gamma_1, \ldots, \gamma_p; \delta_1, \ldots, \delta_p\}$ and if $\{\theta_1, \ldots, \theta_p\}$ is the dual basis for the holomorphic differentials on X, then we can associate with this data the Riemann theta function, θ, whose value at the point $z \in \mathbf{C}^p$ is $\theta(z) = \theta(z, \pi)$, where π is the matrix whose (i,j)-entry is $\pi_{ij} = \int_{\delta_j} \theta_i$. For the theory of this function, we refer the reader to [1, Chapter VI]. In particular, the zeros of the multivalued function on the surface X (here e is an arbitrary point in $\mathbf{C}^p$) $y \mapsto \theta(\Phi_{x_0}(y) - e)$ have been studied extensively.

THEOREM 3. *Let α be a positive integer and let $e \in \mathbf{C}^p$. The multivalued function $y \mapsto \theta(\alpha\Phi_{x_0}(y) - e)$ has precisely $\alpha^2 p$ zeros on X (counting multiplicity) when it does not vanish identically. In this case, the divisor of zeros $y_1 \cdots y_{\alpha^2 p}$ satisfies the equation*

$$\alpha e = \Phi_{x_0}(y_1 \cdots y_{\alpha^2 p}) + \alpha^2 K_{x_0}.$$

The multivalued function vanishes identically if and only if $i[x_0^\alpha \Delta] > 0$, where Δ is an integral divisor of degree $p - 1$ chosen so that $e = \Phi_{x_0}(\Delta) + K_{x_0}$.

PROOF. The proof of the first half of the theorem for $\alpha = 1$ is well known (see, for example, [1, Chapter VI]) and since the same proof works for an arbitrary positive integer α, we leave this as an exercise for the reader. We therefore only give the proof of the second half.

Assume that the function vanishes identically on X. It thus follows that for each point y on X, $\alpha\Phi_{x_0}(y) - e = -\Phi_{x_0}(D(y)) - K_{x_0}$, where $D(y)$ is an integral divisor of degree $p - 1$ on X. Moreover, $e = \Phi_{x_0}(\Delta) + K_{x_0}$ for some integral divisor Δ of degree $p - 1$, since e itself must be a zero of the function. We thus have $\alpha\Phi_{x_0}(y) - \Phi_{x_0}(\Delta) - K_{x_0} = -\Phi_{x_0}(D(y)) - K_{x_0}$. In other words, we have $\Phi_{x_0}(y^\alpha D(y)) = \Phi_{x_0}(\Delta) = \Phi_{x_0}(x_0^\alpha \Delta)$.

It is a consequence of Abel's theorem that the divisors $y^\alpha D(y)$ and $x_0^\alpha \Delta$ are equivalent. Thus for an arbitrary point $y \in X$ not in the support of $x_0^\alpha \Delta$, there is an $f \in L[1/x_0^\alpha \Delta]$ that vanishes at y to order at least α. Since for a β-dimensional space of meromorphic functions on X the possible orders of the functions at most points y are $0, 1, \ldots, \beta - 1$, we conclude that $r[1/x_0^\alpha \Delta] \geq \alpha + 1$ and thus by Riemann-Roch that $i[x_0^\alpha \Delta] \geq 1$. Since the argument is clearly reversible, we have concluded the proof of the theorem.

COROLLARY 1. *If $\alpha \geq p$, then for no point e and no base point x_0 is it the case that $Q \mapsto \theta(\alpha\Phi_{x_0}(Q) - e)$ vanishes identically on X.*

PROOF. If $\alpha \geq p$, then $i[x_0^\alpha \Delta] = 0$ for every integral divisor Δ on X of degree $p - 1$.

Consider the case $\alpha = d = p + k - 1$, $e = h_\rho + \frac{1}{2}\Phi_{x_0}(D_\rho) - K_{x_0}$. According to our theorem and corollary, the multivalued function we are studying has $(p + k - 1)^2 p$ zeros on X. We claim that these are precisely the images under ρ of the points $Q_1, \ldots, Q_{d^2 p}$ listed in Theorem 1.

Recall that the points Q_j in question are the points for which there exists a holomorphic differential in $\mathfrak{A}^-$ with divisor

$$Q_j^{p+k-1}E(Q_j)^{p+k-1}\Delta_{Q_j}E(\Delta_{Q_j}),$$

with Δ_{Q_j} an integral divisor of degree $p-1$. In Lemma 5 we characterized the image of these points under ρ. We can now refine our characterization.

THEOREM 4. *Let S be a compact Riemann surface of genus $g = 2p + k - 1$ which is a two-sheeted branched cover ρ of X, a compact Riemann surface of genus $p > 0$, with ramification divisor $D = x_1 \cdots x_{2k}$. Let $P_i = \rho^{-1}(x_i)$, $i = 1, \ldots, 2k$. Furthermore, let $h = h_\rho$ be the point of order two in $J(X)$ associated with the cover. Let $D_{W^-} = (P_1 \cdots P_{2k})^{d(d-1)/2}Q_1 \cdots Q_{d^2p}E(Q_1) \cdots E(Q_{d^2p})$ be the divisor of Weierstrass points of $\mathfrak{A}^-$. Then the multivalued function*

$$y \mapsto \theta(d\Phi_{x_0}(y) - \tfrac{1}{2}\Phi_{x_0}(x_1 \cdots x_{2k}) + K_{x_0} + h)$$

does not vanish identically. A point $y \in X$ is a zero of this function if and only if $y = \rho(Q_i)$ for some $i = 1, \ldots, d^2\rho$.

PROOF. First note that according to Theorem 3 and its corollary, the expression does not vanish identically on X provided $k > 0$. Assume that $k > 0$. We proceed to identify the zeros of the function.

The Riemann vanishing theorem gives that y is a zero of the function if and only if $d\Phi_{x_0}(y) - \frac{1}{2}\Phi_{x_0}(x_1 \cdots x_{2k}) + K_{x_0} + h = -\Phi_{x_0}(\Delta(y)) - K_{x_0}$, where $\Delta(y)$ is an integral divisor of degree $p - 1$ which depends on y. It thus follows that $\Phi_{x_0}(y^d\Delta(y)) = \frac{1}{2}\Phi_{x_0}(x_1 \cdots x_{2k}) - 2K_{x_0} + h$. Hence y is indeed some $\rho(Q_i)$.

Assume that $k = 0$ and the function vanished identically. The above argument shows that for each $y \in X$, there would exist an integral divisor $\Delta(y)$ of degree $p - 1$ so that $\Phi_{x_0}(y^{p-1}\Delta(y)) = -2K_{x_0} + h_\rho$. This would give that $i[y^{p-1}\Delta(y)] = 1$ (by an argument similar to the one used in the proof of Theorem 3) which is a contradiction to the above equation since $h_\rho \neq 0$.

It is interesting to note that with a little more work we can get a more interesting result. According to Theorem 3, the zero set $\{y_i; i = 1, \ldots, d^2p\}$ of the theta function under consideration satisfies

$$\Phi_{x_0}(y_1 \cdots y_{d^2p}) = \frac{d}{2}\Phi_{x_0}(x_1 \cdots x_{2k}) - dh + \frac{d(d+1)}{2}(-2K_{x_0}).$$

Let us first consider the case of even d. In this case, since $d^2p - kd = d(d+1)(p-1)$, we see quite easily that $y_1 \cdots y_{d^2p}/(x_1 \cdots x_{2k})^{d/2}$ is the divisor of a meromorphic $d(d+1)/2$ differential. Its lift to S is therefore a $d(d+1)/2$ differential and its divisor on S is seen to be $Q_1 \cdots Q_{d^2p} E(Q_1) \cdots E(Q_{d^2p}) \times (P_1 \cdots P_{2k})^{d(d-1)/2}$, where $Q_i \in S$ is chosen so that $\rho(Q_i) = y_i$. If we take the case d odd, then we obtain an anti-invariant $d(d+1)/2$ differential on S with the same divisor as in the even case. We leave the details for the reader.

PROBLEM. From the multivalued function on X, given in the statement of Theorem 4, we have constructed the divisor of a holomorphic $d(d+1)/2$ differential on S. The Wronskian W^- is also such a differential. Both of these differentials vanish at the same points. However, we have not been able to show that the orders of the zeros at a given point coincide (except in special cases).

7. Two-sheeted covers of tori

In this section we refine our previous results for the special case $p = 1$. Thus S is a compact Riemann surface of genus $g \geq 2$ which is a branched two-sheeted cover ρ of a torus X. The surface S then has an involution E with $2g - 2$ fixed points $P_1, \ldots, P_{2g-2}$.

LEMMA 8. (a) *Let P be a fixed point of E. Then both $2g - 4$ and $2g - 2$ cannot occur in the sequence of orders of zeros at P of elements of $\mathfrak{A}^-$. Hence the sequence of zeros is obtained by eliminating either $2g - 4$ or $2g - 2$ from*

$$\{0,2,\ldots,2g-4,2g-2\}.$$

If $g = 2$ or $g = 3$ and S is hyperelliptic, then $2g - 2$ does not appear in the sequence.

(b) *Let Q be a special Weierstrass point for $\mathfrak{A}^-$ that is not a fixed point of E. Then the sequence of orders of zeros at P of elements of $\mathfrak{A}^-$ must contain $g - 1$ and cannot contain $g - 2$ and hence consists of*

$$\{0,1,\ldots,g-3,g-1\}.$$

PROOF. For (a) let us consider the case of a hyperelliptic surface of genus 3 (with hyperelliptic involution H). If the anti-invariant differential ω vanished at P to order 4, it would also vanish to the same order at $(H \circ E)(P)$; obviously impossible (recall that $H \circ E$ is fixed point free). If S has genus 2, then "the" E-anti-invariant differential is $(H \circ E)$-invariant (H, as before, the hyperelliptic involution) and hence vanishes at the two fixed points of $H \circ E$ which are interchanged by E. The rest of the lemma consists of restatements of earlier results.

LEMMA 9. (a) *The weight of a fixed point P of E with respect to the space of anti-invariant differentials is either $(g-1)(g-2)/2$ or $(g^2 - 3g + 6)/2$. The* *former weight occurs when* P^{2g-2} *is not canonical and the latter occurs when* P^{2g-2} *is canonical.*

(b) *The weight of a Weierstrass point* Q *that is not a fixed point of* E *with respect to the space of anti-invariant differentials is* 1.

(c) *Let* $Q \in S$ *be arbitrary. Then* Q^{2g-2} *is canonical if and only if* $Q^{2g-2}/P_1P_2\cdots P_{2g-2}$ *is principal.*

PROOF. Left to the reader.

What are the classical Weierstrass points (with respect to $\mathfrak{A}$) on S? We should be able to describe them in terms of quantities on the torus X. The Wronskian W with respect to all the differentials on S is a holomorphic $g(g+1)/2$ differential. This differential is invariant if g is even and anti-invariant if g is odd. Its zeros are the Weierstrass points of S. Let $\nu_j = \operatorname{ord}_{P_j} W$, $j = 1,\ldots,2g-2$. It is easy to show (since the invariant differential has a simple zero at P) that $\nu_j = (g-3)(g-2)/2$ or $\nu_j = 2 + (g-3)(g-2)/2$. The lower value always applies when S is hyperelliptic and hence $g = 3$ or $g = 2$. Let $Q_1,\ldots,Q_n,E(Q_1),\ldots,E(Q_n)$ be the other zeros of W listed according to their multiplicities. Then $\sum_{j=1}^{2g-2}\nu_j + 2n = g(g^2-1)$.

Assume now that g is odd. The projection $\rho(W)$ of W to X has order $\mu_j = \nu_j/2 - g(g+1)/4$ at $\rho(P_j)$, $j = 1,\ldots,2g-2$ and vanishes at $\rho(Q_1),\ldots,\rho(Q_n)$. We note that the possible values of μ_j are $3(1-g)/2$ and $(5-3g)/2$. (These are (negative) integers only when g is odd, as expected.) We conclude that the integral divisors $\prod_{j=1}^{2g-2}\rho(P_j)^{-\mu_j}$ and $\prod_{j=1}^{n}\mu(Q_j)$ have the same degree (n) and are equivalent.

Assume next that g is even. Then W^2 is an invariant differential and the corresponding analysis with $\mu_j = \operatorname{ord}_{\rho(P_j)}\rho(W^2)$ shows that $\prod_{j=1}^{2g-2}\rho(P_j)^{-\mu_j}$ and $\prod_{j=1}^{n}\rho(Q_j)^2$ are equivalent divisors. The only possible values for μ_j are $3-3g$ and $5-3g$.

REMARK. The above relations may also be derived by projecting the functions $W/\theta^{g(g+1)/2}$ for g odd and $W^2/\theta^{g(g+1)}$ for g even to X, where θ is a non-trivial E-invariant differential on X.

As an application we see that for genus 2, the six Weierstrass points on S can be paired: $Q_1,E(Q_1)$, $Q_2,E(Q_2)$, $Q_3,E(Q_3)$. The product of the cubes of the two branch values $\rho(P_1)^3\rho(P_2)^3$ is equivalent to the product of the squares of the images of the Weierstrass points $\rho(Q_1)^2\rho(Q_2)^2\rho(Q_3)^2$. As another application we can consider the case of a hyperelliptic surface of genus 3, where we find that the Weierstrass points come in pairs $Q_j,E(Q_j)$, $j = 1,\ldots,4$, and the product of the cubes of the four branch values $\prod_{j=1}^{4} \rho(P_j)^3$ is equivalent to the product of cubes of the image of the Weierstrass points $\prod_{j=1}^{4} \rho(Q_j)^3$.

The Weierstrass points for $\mathfrak{A}^-$ are particularly easy to describe in the case we are considering.

PROPOSITION 2. *Choose a special Weierstrass (for $\mathfrak{A}^-$) point $R_1 \in S$. Let us normalize the torus X so that $\rho(R_1) = 0$, the origin. Let $\{R_1, \ldots, R_m; 0 < m \leq 2(g-1)^2\}$ be the lifts to S of the points of order $g-1$ on the torus. Each R_i in this list is a Weierstrass point for the space of anti-invariant differentials. It is a simple Weierstrass point if it is not a fixed point of E; otherwise it is a point of weight $(g-1)(g-2)/2 + 2$. The fixed points of E not in the above list are the remaining Weierstrass points, each of weight $(g-1)(g-2)/2$.*

PROOF. The special Weierstrass points for $\mathfrak{A}^-$ are precisely those points Q in S for which there exists a $\omega_Q \in \mathfrak{A}$ with $(\omega_Q) = Q^{g-1}E(Q)^{g-1}$. Such an ω_Q is automatically in $\mathfrak{A}^-$. Now the E-invariant function ω_Q/ω_{R_1} projects to a function on the torus with divisor $\rho(Q)^{g-1}/\rho(R_1)^{g-1}$. Hence if $\rho(R_1) = 0$, we see that $\rho(Q)$ is a point of order $g-1$ on X. Conversely, every point of order $g-1$ on X lifts to a special Weierstrass point on S. The remaining Weierstrass points of $\mathfrak{A}^-$ must be fixed points for E and not special.

REMARKS. (1) It is of some interest to compare this last result with the situation in Theorem 4. The normalization chosen $\rho(R_1) = 0$ for the torus is equivalent to choosing $x_0 = \rho(Q_1)$. Thus the origin is a zero of the multivalued function. We therefore have

$$\theta(-\tfrac{1}{2}\Phi_{x_0}(x_1 \cdots x_{2k}) + K_{x_0} + h_\rho) = 0.$$

In genus 1, the theta function has exactly one zero at the point $(1+\tau)/2 = K_{x_0}$ for every choice of x_0. Hence we can conclude that $-\frac{1}{2}\Phi_{x_0}(x_1 \ldots x_{2k}) + h_\rho = 0$ where $x_0 = \rho(Q_1)$ and Q_1 is a special point.

(2) Consider the case of odd genus g. Each point R_i gives rise to a canonical divisor $R_i^{g-1}E(R_i)^{g-1}$ and thus to the half canonical divisor $R_i^{(g-1)/2}E(R_i)^{(g-1)/2}$. These half canonical divisors give rise to points of order two in $J(S)$,

$$e_i = \Phi_{x_0}(R_i^{(g-1)/2}E(R_i)^{(g-1)/2}) + K_{x_0}.$$

These points of order 2 are independent of the choice of base point x_0 and the theta function vanishes at these points of order 2 to even or odd order depending on the index of specialty of the half canonical divisor used.

(3) Let us choose the indices so that

$$\rho(R_2)^{(g-1)/2}, \quad \rho(R_3)^{(g-1)/2} \quad \text{and} \quad \rho(R_4)^{(g-1)/2}$$

are the 3 points of order 2 on X (in addition to the origin). It follows that $(R_1R_2R_3R_4)^{(g-1)/2}E(R_1R_2R_3R_4)^{(g-1)/2}$ is the divisor of a holomorphic quadratic differential on S.

8. Special choices for the branch points

In this section we study some special two-sheeted covers of tori which we construct by properly choosing the branch values of the cover. In this way we shall determine what it means for a fixed point of E to be a special Weierstrass point of the space of anti-invariant differentials (hence of weight $(g^2 - 3g + 6)/2$). We shall be able to determine which covers of genus 3 are hyperelliptic and how to construct these surfaces from the torus.

Let X be a torus with a fixed point $x_0 \in X$ corresponding to the origin in the usual representation of X as the plane factored by the lattice generated by 1 and the point τ in the upper half plane. The torus X has a unique involution ψ fixing $x_0 = 0$. It is given by $z \mapsto -z$. The other 3 fixed points of ψ are the half periods $1/2$, $\tau/2$ and $(1+\tau)/2$. For $g \geq 2$, let us now choose $2g-2$ points on the torus by first selecting $g-1$ distinct points x_i, $i = 1, \ldots, g-1$, none of these of order 2 (that is, none fixed by ψ) and their images under the involution $x_i = \psi(x_{i-g+1})$, $i = g, \ldots, 2g-2$.

Consider now a two-sheeted cover of X branched over these $2g-2$ points. There are in fact four possible covers which we can construct. Fix one such cover, which we shall denote by S. We already know that S carries a conformal involution E with $S/\langle E\rangle = X$ and E has $2g-2$ fixed points which project to the points x_i, $i = 1, \ldots, 2g-2$, on X. We claim that the involution ψ of X lifts to S. To verify this assertion, we study the action of ψ on the fundamental group of X punctured at the $2g-2$ points x_i.

Let $\mathcal{P}$ be the period parallelogram for the torus X with vertices

$$\{-1/2 - \tau/2, 1/2 - \tau/2, 1/2 - \tau/2, -1/2 + \tau/2\}.$$

We let the origin correspond to the base point of the fundamental group of the punctured torus (the punctures are points in $\mathcal{P}$). Let γ and δ be curves through the origin that are invariant under ψ, avoid all the punctures, and project to a canonical homology basis on the (unpunctured) torus. It is easy to choose such curves.

We may and do assume that γ (respectively, δ) runs from $-1/2$ ($-\tau/2$) to $1/2$ ($\tau/2$). For $i = 1, \ldots, g-1$, choose the curves c_i as in §2. For $i = g, \ldots, 2g-2$, we let $c_i = \psi(c_{i-g+1})$. The $2g$ curves we have constructed generate the fundamental group of the punctured surface.

Let h be the defining homomorphism from the fundamental group of the punctured torus into $\mathbf{Z}_2$ for the cover $\rho: S \to X$. Recall that $h(c_i) = 1$ for each i and that h is arbitrary on γ and δ. Note $\psi(\gamma) = \gamma^{-1}$ and $\psi(\delta) = \delta^{-1}$. Hence $h \circ \psi = -h$ and the involution ψ lifts to every cover S. Let Ψ be a lift of ψ to S. Since $\psi \circ \rho = \rho \circ \Psi$ and $\rho \circ E = \rho$ both $E \circ \Psi$ and $\Psi \circ E$ are also lifts of ψ and $E \circ \Psi = \Psi \circ E$. Now if Q is a fixed point of Ψ, then $\rho(Q)$ is a fixed point of ψ. It follows that the involution Ψ has 0, 2, 4, 6 or 8 fixed points. To be specific, let us choose a point $Q_0 \in S$ with $\rho(Q_0) = 0$ and let us choose Ψ to satisfy $\Psi(Q_0) = Q_0$. Thus Ψ fixes the two points over the origin (and $\Psi \circ E$ interchanges them). To see whether the two points above another half period, say x, are fixed by Ψ, we join the origin to the point x by a smooth curve in the punctured torus. Then this curve followed by its image under ψ is a closed curve through the origin on the punctured torus. The two points lying over x are fixed by Ψ if and only if the curve we have constructed is in the defining subgroup of the cover. It thus follows that the two points lying over $1/2$ are fixed by Ψ (and hence interchanged by $\Psi \circ E$) if and only if $h(\gamma) = 0$, and the two points lying over $\tau/2$ are fixed by Ψ if and only if $h(\delta) = 0$. A curve from $-(\tau + 1)/2$ to $(\tau + 1)/2$ through the origin and invariant under ψ may be chosen so that it is homotopic to γ followed by δ followed by one half $(g-1)$ of the c_i. Hence the two points lying over $(\tau + 1)/2$ are fixed provided $h(\gamma) + h(\delta) + g - 1 = 0$. Hence for g odd when $h(\gamma) = h(\delta)$ and for g even when $h(\gamma) \neq h(\delta)$. We summarize our construction in

LEMMA 10. (a) *If g is even, then Ψ has* 2 *or* 6 *fixed points. The number of fixed points for the corresponding map $\Psi \circ E$ are* 6 *and* 2.

(b) *If g is odd, then Ψ has* 4 *or* 8 *fixed points. The number of fixed points for the corresponding map $\Psi \circ E$ are* 4 *and* 0.

(c) *For $g = 2$ and $g = 3$, the surface S is hyperelliptic if and only if Ψ has the maximum number of fixed points* (6 *for genus* 2 *and* 8 *for genus* 3) *in which case it is the hyperelliptic involution.*

In the above construction our only condition on the points x_i was that none be of order 2. The points may be chosen to be of order $(g-1)$ and not of order 2 provided, of course, that $g > 3$. In this case, we conclude that the lifts of these points would be fixed by E and also special Weierstrass points for $\mathfrak{A}_E^-$. If we give up the ability to lift the involution ψ of X to S, then we can allow the x_i to be points of order 2. Consider the case of genus 3. The 4 bitangents correspond to 4 pairs of special points on S. By properly choosing the branch values on X, each such special pair may or may not correspond to a degenerate bitangent (by varying the complex structure, the two special points not fixed by E may be forced to coalesce to a single fixed point of E). We have returned to the situation that motivated this work. It is a good place to end this chapter.

Note added in proof (April 15, 1991). After compilation of this paper one of the authors recalled a beautiful book that both authors had previously read with great pleasure. The reader is asked to consult §5.2 (entitled "Why twenty-eight bitangents") of C. Herbert Clemens' book *A Scrapbook of Complex Curve Theory*, Plenum Press, New York, 1980) for a further introduction to the problem that motivated this work.

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